Fibonacci diagonals otherwise

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Part of a Discrete Maths exam question asks for a proof by induction of the result,

$$\sum_{0\le k\le n} \binom{n-k}{k} = F_{n+1},$$

where F_n denotes the n'th Fibonacci number. If the question had said, "by induction *or otherwise*," the following argument using generating functions would have been apt.

Let s_n be the sum in the question, and let $G(t)$ be the generating function,

$$G(t) = \sum_{n\ge 0} s_n t^n.$$

We will show that $G(t) = 1/(1-t-t^2)$, agreeing with the well-known generating function for the sequence $\langle F_{n+1}\rangle$.

Substituting in the definition of s_n and exchanging the order of summation gives

$$G(t) = \sum_{k\ge 0}\sum_{n\ge k} \binom{n-k}{k} t^n.$$

Observe that if $n < 2k$ then the binomial coefficient is zero; so we can refine the bounds of the inner sum and take out a factor of t^{2k}.

$$G(t) = \sum_{k\ge 0}\left(t^{2k} \sum_{n\ge 2k} \binom{n-k}{k} t^{n-2k} \right).$$

Now we shift the bounds of summation, and "negate the upper index," using the identity,

$$\binom{n+k}{k} = \binom{n+k}{n} = (-1)^n \binom{-(k+1)}{n},$$

giving

$$G(t) = \sum_{k\ge 0}\left(t^{2k} \sum_{n\ge 0} \binom{-(k+1)}{n} (-t)^n \right).$$

But this inner sum is just the binomial expansion of $(1-t)^{-(k+1)}$, so we obtain

$$G(t) = \frac{1}{1-t}\sum_{k\ge 0}\left(\frac{t^2}{1-t}\right)^k.$$

Applying the binomial theorem again gives

$$G(t) = \frac{1}{(1-t)\left(1 - \frac{t^2}{1-t}\right)} = \frac{1}{1-t-t^2},$$

as required.

To verify the claim that this is the generating function for the sequence $\langle F_{n+1} \rangle$, we use the recurrence $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$, with $F_1 = F_2 = 1$. This gives

$$G(t) = 1 + t + \sum_{n \geq 2} (F_n + F_{n-1}) t^n.$$

Thus, shifting the index of summation appropriately,

$$G(t) = 1 + t\left(1 + \sum_{n \geq 1} F_{n+1} t^n\right) + t^2 \sum_{n \geq 0} F_{n+1} t^n = 1 + tG(t) + t^2 G(t),$$

so $(1 - t - t^2)G(t) = 1$.